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Congruences of Dislocations in Continuously Dislocated Crystals

A. Trzęsowski[1]

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The time-dependent congruences of Volterra-type dislocations are investigated based on the generalized formulas of Frenet in a Riemannian space. The analysis is applied to the description of congruences of edge and mixed dislocations consistent with a continuous distribution of dislocations for which its material space is an equidistant Riemannian space. In particular, the principal congruences of dislocations are considered. The kinematics of congruences of mixed dislocations endowed with univocally defined local slip planes is discussed. It is shown that the geometry of such congruences of dislocations admits a class of nonlinear evolution equations describing the curvature and torsion of a congruence of curves in a Riemannian space. Additional conditions imposed on the derived system of equations in order to describe the evolution of curvature and torsion of congruences of edge dislocations are proposed. In the static case, an expression is given for shear stresses required to bend prismatic edge dislocations of torsion zero located on the totally geodesic crystal surfaces. It follows that the congruence of these dislocations is endowed with a finite self-energy function.

1. INTRODUCTION

There are two basic types of dislocation movement, *glide*, in which the dislocation moves in a surface, called the *slip surface*, which contains its line and Burgers vector, and *climb*, in which the dislocation moves out of this surface normal to the Burgers vector (Hull and Bacon, 1984). For example, a straight *edge* dislocation has a rigorously defined *slip plane* in which it can move. The plane includes the dislocation and its Burgers vector orthogonal to the dislocation line. Likewise, when a Burgers vector is not in the plane of a flat, curved edge dislocation line, the dislocation has a rigorously defined *slip surface* in which the dislocation can glide. The dislocation is then called a *prismatic dislocation*. For example, a prismatic edge dislocation loop can move only by glide on a cylindrical surface, and if the loop expands or shrinks, climb must be occuring. There are also

[1] Institute of Fundamental Technological Research, Polish Academy of Sciences, 00-049 Warsaw, Poland.

prismatic dislocations in the form of cylindrical helices. Namely, dislocations in the form of a long spiral have been observed in crystals (Hull and Bacon, 1984). The spiral dislocation lies on a cylinder whose axis is parallel to the Burgers vector, and the dislocation can glide on this cylinder. Consequently, the *prismatic helical dislocation* is *mixed* (with edge and screw components; see hereafter and Section 2). The planes tangent to the slip surface of a prismatic dislocation are *local slip planes*. The Burgers vector of prismatic dislocations, edges as well as mixed cylindrical helices, is parallel to the local slip planes. Note that the Burgers vector of straight *screw dislocation* is parallel to the dislocation line and thus the glide of this dislocation is not restricted to a specific plane.

The *slip*, which is the most common manifestation of plastic deformation in crystalline solids, can be envisaged as sliding or successive displacement of one plane of atoms over another on a distinguished plane called the (local or global) *slip plane*. Discrete blocks of crystal between two slip planes remain undistorted (Hull and Bacon, 1984). Consequently, any dislocation line in the crystal can be treated as a line formed by means of a slip (homogeneous or not) such that the dislocation becomes a boundary between the slipped and unslipped parts of the crystal (Hull and Bacon, 1984; Fridmann, 1974). The *slip direction* is then parallel to the Burgers vector of the dislocation, and the *slip magnitude* equals the strength of dislocation. If we deal with a prismatic dislocation, then the slip is called *prismatic*. The above representation of a dislocation concerns flat as well as spatial dislocation lines (Fridmann, 1974) and the dislocations so represented are called *Volterra dislocations* (Hull and Bacon, 1984). On the other hand, it is known that the glide motion of many dislocations results in slip, and it is observed that globally (i.e., on a macroscale) this motion is accompanied by the occurrence of *slip surfaces* (Hull and Bacon, 1984). Therefore, we can generalize the notion of line defects of a crystal structure by defining a *dislocation line* in a continuously dislocated crystal as a boundary between slipped and unslipped parts of the crystal located on a slip surface. The so-defined *Volterra-type* dislocation line can be endowed, in the continuous approximation, with the so-called *local Burgers vector* (Trzęsowski, 1994) tangent to the slip surface along the line everywhere. Thus, a *resulting Burgers vector* of the dislocation can be defined (Trzęsowski, 1994). The glide motion of such an "effective" dislocation can be considered as a *mesoscopic* elementary act of macroplasticity (cf. Trzęsowski, 1997). More generally, we can extend this definition of dislocation lines on each curve (flat or spatial) that can be endowed with the local Burgers vector in a manner consistent with the considered continuous distributions of dislocations (Section 2). Later we consider, in general, dislocation lines understood in this broader sense.

The occurrence of many dislocations in a Bravais crystal structure generates a bend of originally straight lattice lines of this crystal structure (Orlov, 1983). Consequently, the *lattice lines* in a continuously dislocated *Bravais crystal* form a system of three independent congruences of curves, and tangents to these curves define *local crystallographic directions* of the continuized Bravais crystal with many dislocations. Planes spanned by two local crystallographic directions are *local crystal planes*. In general, none of these congruences is normal (i.e., the curves of the congruence are not orthogonal trajectories of a family of surfaces). If a *crystallographic congruence*, that is, a congruence of lattice lines, is normal and its curves are orthogonal to local crystal planes everywhere, then the curves are orthogonal trajectories of a family of *crystal surfaces* of the continuously dislocated Bravais crystal. The mean value of normal curvatures κ_n of a crystal surface in its local crystallographic directions (see, e.g., Eisenhart, 1964) can be, for example, approximated by (Orlov, 1983)

$$\kappa_n = \rho b, \quad [\kappa_n] = \mathrm{cm}^{-1}, \quad [\rho] = \mathrm{cm}^{-2}, \quad [b] = \mathrm{cm}, \tag{1.1}$$

where ρ denotes the (mean) density of dislocations defined as the length of all dislocation lines included in the volume unit, and b is the mean strength of dislocations. If, additionally, the local crystal planes are local slip planes for a congruence of dislocations, the crystal surfaces are virtually slip surfaces for dislocations of this congruence. Such slip surfaces will also be called *glide surfaces*. In particular, in the case of *single glide*, crystal planes originally parallel and normal to a lattice direction pass into the glide surfaces without local stretchings (Bilby *et al.*, 1958) and thus the crystal surfaces must be flat.

The occurrence of many dislocations in a Bravais crystal structure is accompanied by the existence of *secondary point defects* of this crystal structure created by the distribution of dislocations. It is, for example, due to intersections of dislocation lines: point defects can appear at crossover points of edge dislocation lines or when two parallel dislocation lines join together (Oding, 1961). On the other hand, dislocations have no influence on local metric properties of a crystal structure (since a crystal with many dislocations can be approximately considered locally as part of an ideal crystal—Trzęsowski, 1993). Consequently, the influence of secondary point defects on the metric properties of a continuously dislocated Bravais crystal can be modeled by the assumption that the considered body is additionally endowed with a Riemannian metric that reduces to a Euclidean metric when dislocations are absent (Trzęsowski, 1994, 1995, 1997). The occurrence of secondary point defects influences the geometry of crystal and slip surfaces as well as congruences of dislocation lines. It can be described by means of the treatment of dislocation lines, understood in the above-defined generalized sense, local crystal planes, and local slip planes as those located in the above-defined *Riemannian material space* (Sections 2 and 3). In particular, the geometry of congruences of dislocation lines can be described based on the generalized formulas of Frenet in a Riemannian space (Sections 5 and 6). The analysis is applied to the description of congruences of dislocations consistent with a continuous distribution of dislocations for which its material space is an equidistant Riemannian space (Sections 3–6).

The *plastic flow* in crystals with many dislocations is accompanied by the motion of congruences of dislocations (Trzęsowski, 1997, 2000). Therefore, it is reasonable to study various kinematic properties of the motion of these congruences. The kinematic properties investigated in the paper concern congruences of time-dependent mixed dislocations endowed with the univocally defined local slip planes (Sections 2 and 7). The geometry of such congruences of dislocations admits a class of nonlinear evolution equations describing the curvature and torsion of a congruence of curves in a Riemannian space (Section 7). Additional conditions imposed on the derived system of equations in order to describe the evolution of curvature and torsion of congruences of edge dislocations are proposed (Section 7). It is given, in the static case, an expression for shear stresses required to bend prismatic edge dislocations of torsion zero located on the totally geodesic crystal surfaces. It follows that the congruence of these dislocations must be endowed with a finite self-energy function (Section 8).

2. LOCAL GLIDE SYSTEMS AND SLIP PLANES

Let $\mathcal{B}$ be a body identified with its distinguished spatial configuration being an open and contractible to a point subset of the Euclidean configurational point E^3 of the body (Trzęsowski, 1993). Let $\Phi = (\boldsymbol{E}_a;\ a = 1, 2, 3)$, $[\boldsymbol{E}_a] = \mathrm{cm}^{-1}$, be a dimensional base of smooth vector fields on $\mathcal{B}$. Later, we consider dimensional coordinate systems $X = (X^A; A = 1, 2, 3)$, $[X^A] = \mathrm{cm}$, on $\mathcal{B}$. Then

$$\boldsymbol{E}_a = \underset{a}{e}^A \partial_A, \quad \partial_A = \frac{\partial}{\partial X^A}$$
$$[\partial_A] = \mathrm{cm}^{-1}, \quad \underset{a}{e}^A \in C^\infty, \quad [\underset{a}{e}^A] = [1], \tag{2.1}$$

and the cobase $\Phi^* = (E^a; a = 1, 2, 3)$ dual to Φ has the following representation:

$$E^a = \overset{a}{e}_A dX^A, \quad [E^a] = [dX^A] = \mathrm{cm},$$
$$\langle E^a, \boldsymbol{E}_b \rangle = \overset{a}{e}_A \underset{b}{e}^A = \delta^a_b. \tag{2.2}$$

The object of *anholonomity* (C^a_{bc}) of Φ is then given by

$$[\boldsymbol{E}_a, \boldsymbol{E}_b] = \boldsymbol{E}_a \circ \boldsymbol{E}_b - \boldsymbol{E}_b \circ \boldsymbol{E}_a = C^c_{ab} \boldsymbol{E}_c,$$
$$C^c_{ab} \in C^\infty, \quad [C^c_{ab}] = \mathrm{cm}^{-1} \tag{2.3}$$

If the object of anholonomity does not vanish identically, then the base Φ can be considered as defining a system of three independent *crystallographic congruences* of a continuously dislocated Bravais crystal and thus describing a bend, due to the occurrence of many dislocations, of originally straight lattice lines (Section 1). The base Φ then is called a *Bravais moving frame*. The object of anholonomity of a Bravais moving frame represents the *long-range distortion* of a continuously dislocated Bravais crystal.

The influence of secondary point defects (Section 1) on the metric properties of the continuously dislocated Bravais crystal is described by the following *intrinsic* *metric tensor* (Trzęsowski, 1994):

$$\boldsymbol{g} = \delta_{ab} E^a \otimes E^b = g_{AB} dX^A \otimes dX^B,$$
$$g_{AB} = \overset{a}{e}_A \overset{b}{e}_B \delta_{ab}, \quad [\boldsymbol{g}] = \mathrm{cm}^2. \tag{2.4}$$

The Riemannian space $\mathcal{B}_g = (\mathcal{B}, \boldsymbol{g})$ is a *material space* associated with the considered distribution of dislocations. Note that we can introduce, in a neighborhood of each point $p \in \mathcal{B}$, the so-called normal Riemannian coordinates $\xi = (\xi^a;\ a = 1, 2, 3)$ (Eisenhart, 1964) such that $E^a(p) = d\xi^a_p$ in (2.4). It describes the property of continuously dislocated crystals that dislocations have no influence on the metric properties of an infinitesimal material neighborhood identified, in the continuous limit, with a macroscopically small homogeneous neighborhood of the crystalline body point (Trzęsowski, 2000). This property can be extended on a finite material neighborhood of each body point iff the Riemannian material space is flat. Note that the flatness of the space $\mathcal{B}_g$ does not mean a lack of dislocations (see Section 3). Therefore, we consider the base vector fields of a Bravais moving frame as those defining the local crystallographic directions (Section 1) as well as local scales of an internal length measurement along these directions. It is a representation of the *short-range order* of a continuously dislocated crystal.

Now, we can represent the influence of secondary point defects on the distribution of dislocations (Section 1) by means of the treatment of the Bravais moving frame $\Phi = (\boldsymbol{E}_a)$ and its object of anholonomity (C^c_{ab}) as geometric objects defined on the Riemannian material space B_g. This means that the base vector fields $\boldsymbol{E}_a$, $a = 1, 2, 3$, are considered as orthonormal,

$$\boldsymbol{E}_a \cdot \boldsymbol{E}_b = \underset{a}{e}^A \underset{b}{e}^B g_{AB} = \delta_{ab}, \tag{2.5}$$

and the so-called *dislocation density tensor* $\boldsymbol{\alpha}$ is defined by (Trzęsowski, 1993)

$$\boldsymbol{\alpha} = \alpha^{ab} \boldsymbol{E}_a \otimes \boldsymbol{E}_b, \quad [\boldsymbol{\alpha}] = \mathrm{cm}^{-3},$$
$$\alpha^{ab} = -\frac{1}{2} e^{acd} C^b_{cd}, \quad [\alpha^{ab}] = \mathrm{cm}^{-1}, \tag{2.6}$$

where $e^{abc} \overset{*}{=} \varepsilon^{abc}$ denotes the permutation symbol ε^{abc} associated with Bravais moving frame $\Phi = (\boldsymbol{E}_a)$ and considered as a contravariant 3-vector density of weight $+1$ in $\mathcal{B}_g$ (Gołąb, 1966). Likewise, the *scalar volume dislocation density* ρ of a finite total length $L_\mathrm{d}(\mathcal{B})$ of dislocation lines will be measured with respect to the material volume 3-form ω_g:

$$0 < L_\mathrm{d}(\mathcal{B}) = \int_{\mathcal{B}} \rho \omega_g < \infty, \tag{2.7}$$

where

$$\omega_g = E^1 \wedge E^2 \wedge E^3 = e dX^1 \wedge dX^2 \wedge dX^3,$$
$$e = \det\left(\overset{a}{e}_A\right) = g^{1/2}, \quad g = \det(g_{AB}),$$
$$[\rho] = \mathrm{cm}^{-2}, \quad [\omega_g] = \mathrm{cm}^3, \quad [L_\mathrm{d}(\mathcal{B})] = \mathrm{cm}. \tag{2.8}$$

If $\boldsymbol{l} = l^a \boldsymbol{E}_a$, $[\boldsymbol{l}] = \text{cm}^{-1}$, is a unit vector field defining a congruence of dislocation lines in the continuized Bravais crystal (Section 1) and considered as these located in the Riemannian material space $\mathcal{B}_g$ (Section 1), then the *local Burgers vector* $\boldsymbol{b}$ of the congruence is defined by (Trzęsowski, 1994, 1997)

$$\boldsymbol{b} = b^a \boldsymbol{E}_a, \quad [b^a] = \text{cm},$$
$$b_g = \|\boldsymbol{b}\|_g = (b_a b^a)^{1/2} > 0, \quad b_a = \delta_{ac} b^c, \quad [b_g] = \text{cm}, \tag{2.9}$$

where the components b^a of $\boldsymbol{b}$ are defined by

$$\rho b^a = l_b \alpha^{ba}, \quad l_a = \delta_{ab} l^b, \quad l_a l^a = 1, \quad [\rho] = \text{cm}^{-2}, \tag{2.10}$$

and ρ is the scalar volume dislocation density of (2.7). A dislocation line with its unit tangent $\boldsymbol{l}$ and the local Burgers vector $\boldsymbol{b}$ is interpreted as the *edge* dislocation line if

$$\boldsymbol{b} \cdot \boldsymbol{l} = b^a l_a = b_g m^a l_a = 0, \tag{2.11}$$

where

$$\boldsymbol{b} = b_g \boldsymbol{m}, \quad [\boldsymbol{m}] = \text{cm}^{-1},$$
$$\boldsymbol{m} = m^a \boldsymbol{E}_a, \quad \|\boldsymbol{m}\|_g = (m_a m^a)^{1/2} = 1, \tag{2.12}$$

and (2.9) was taken into account. A dislocation line is interpreted as the *screw* dislocation line if

$$\boldsymbol{b} = \eta \boldsymbol{l}, \quad \eta \neq 0, \tag{2.13}$$

In other cases, a dislocation is interpreted as the *mixed* (with edge and screw components) dislocation line.

If the unit vector field $\boldsymbol{l}$ defines a congruence of edge dislocation lines and the vector field $\boldsymbol{b}$ of (2.9)–(2.12) is the local Burgers vector of this congruence, then the family $\pi(\boldsymbol{l}, \boldsymbol{m})$ of planes spanned by the vector fields $\boldsymbol{l}$ and $\boldsymbol{m}$ consists of *local slip planes* (Section 1) of the congruence. The family $\pi(\boldsymbol{l}, \boldsymbol{m})$ is univocally defined by the congruence. Let $\boldsymbol{n}$ be the unit vector field on $\mathcal{B}_g$ normal to the slip planes of the congruence. The ordered triple $(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n})$ is uniquely determined up to its orientation and defines the two-dimensional distribution $\pi_{\boldsymbol{n}}(\boldsymbol{l}, \boldsymbol{m})$ of oriented local slip planes. The ordered triple $(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n})$ and the ordered pair $(\boldsymbol{m}, \boldsymbol{n})$ are called the *local glide system* and the *local slip system* of the congruence, respectively (Trzęsowski, 1997). The local slip system $(\boldsymbol{m}, \boldsymbol{n})$ defines local slip planes as those normal to the $\boldsymbol{n}$ direction and defines the direction $\boldsymbol{m}$ of local slips on these planes (Section 1). This notion is used in plasticity theory, but the notion of local glide system is not considered in this theory. If the oriented two-dimensional distribution $\pi_{\boldsymbol{n}}(\boldsymbol{l}, \boldsymbol{m})$ is (completely) integrable (Sikorski, 1972; Von Westenholz, 1978), then through each point of the body $\mathcal{B}$ there passes a unique maximal integral manifold of the distribution. These integral manifolds are virtually *oriented slip surfaces* in which dislocation lines of the congruence can glide (Section 1).

For screw dislocation lines, the local slip planes are not univocally defined. Let us write, in order to describe congruences of mixed dislocations endowed with univocally defined local slip planes, the dislocation density tensor $\boldsymbol{\alpha}$ of (2.6) in the following form:

$$\alpha^{ab} = \gamma^{ab} + \omega^{ab}, \tag{2.14}$$

where

$$\gamma^{ab} = \alpha^{(ab)}, \quad \omega^{ab} = \alpha^{[ab]} = \frac{1}{2} t_c e^{cba}, \tag{2.15}$$

and

$$t_a = C^c_{ac} = e_{abc}\alpha^{bc}, \tag{2.16}$$

where $e_{abc} \overset{*}{=} \varepsilon_{abc}$ denotes the permutation symbol ε_{abc} (=ε^{abc}) associated with the Bravais moving coframe $\Phi^* = (E^a)$ and considered as a covariant 3-vector density of weight -1 in $\mathcal{B}_g$ (Gołąb, 1966). The object of anholonomity can be written, according to (2.6) and (2.14)–(2.16), in terms of the dislocation density tensor:

$$C^a_{ab} = t_{[\mathrm{b}}\delta^a_{c]} - e_{bcd}\gamma^{da}. \tag{2.17}$$

Therefore, the long-range distortion of the continuously dislocated Bravais crystal characterizes the pair $(\boldsymbol{\gamma}, \boldsymbol{t})$, where

$$\begin{gathered} \boldsymbol{\gamma} = \gamma^{ab}\boldsymbol{E}_a \otimes \boldsymbol{E}_b, \quad \gamma^{ab} = \gamma^{ba}, \\ \boldsymbol{t} = t^a\boldsymbol{E}_a, \quad t^a = \delta^{ab}t_b, \quad [\gamma^{ab}] = [t^a] = \mathrm{cm}^{-1}. \end{gathered} \tag{2.18}$$

Introducing designations

$$\begin{gathered} \boldsymbol{t} = t_g\boldsymbol{s}, \quad \boldsymbol{s} = s^a\boldsymbol{E}_a, \quad \|\boldsymbol{s}\|_g = 1, \\ t_g = \|\boldsymbol{t}\|_g = (t_a t^a)^{1/2}, \quad [\boldsymbol{s}] = [t_g] = \mathrm{cm}^{-1}, \end{gathered} \tag{2.19}$$

and

$$\begin{gathered} \boldsymbol{M} = M^a\boldsymbol{E}_a = M_g\boldsymbol{m}, \quad \|\boldsymbol{m}\|_g = 1, \\ M^a = l_b s_c e^{bca}, \quad s_c = \delta_{ca}s^a, \end{gathered} \tag{2.20}$$

where

$$M_g = \|\boldsymbol{M}\|_g = \sin\varphi, \quad \cos\varphi = \boldsymbol{s}\cdot\boldsymbol{l}, \quad 0 < \varphi < \pi, \tag{2.21}$$

we can write, according to (2.9), (2.10), (2.14), and (2.18)–(2.21), the local Burgers vector $\boldsymbol{b}$ in the form

$$\rho\boldsymbol{b} = \boldsymbol{\gamma}\boldsymbol{l} + \mu\boldsymbol{m}, \quad \boldsymbol{l}\cdot\boldsymbol{m} = 0, \quad \mu = \frac{1}{2}M_g t_g \geq 0, \quad [\mu] = \mathrm{cm}^{-1}. \tag{2.22}$$

It follows from (2.11) and (2.22) that a congruence of dislocation lines tangent to the $\boldsymbol{l}$ direction consists of edge dislocations if

$$\boldsymbol{l}\gamma\boldsymbol{l} = 0, \quad \|\boldsymbol{l}\|_g = 1. \tag{2.23}$$

Moreover, each line in $\mathcal{B}_g$ defines a dislocation line iff

$$\gamma\boldsymbol{t} \neq \boldsymbol{0}, \quad \text{or} \quad \text{rank}\ \gamma = 3. \tag{2.24}$$

For example, if $\gamma\boldsymbol{t} = \boldsymbol{0}$ for $\boldsymbol{t} \neq \boldsymbol{0}$, then $\boldsymbol{l} = \boldsymbol{s}$ does not define a congruence of dislocations. If $\boldsymbol{t} \neq \boldsymbol{0}$ and rank $\gamma = 3$, then the condition (2.23) is fulfilled iff γ is an indefinite nonsingular tensor field with $\boldsymbol{l}$ being its null (isotropic) vector field. Thus, in this case, edge dislocation lines are located on null cones of γ.

Let $(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n})$ be a triple of $\boldsymbol{g}$-orthonormal vector fields defined by (2.22) and the following condition:

$$\boldsymbol{n}\gamma\boldsymbol{l} = 0. \tag{2.25}$$

Then

$$\boldsymbol{b} = b_{(l)}\boldsymbol{l} + b_{(m)}\boldsymbol{m}, \quad \boldsymbol{l} \cdot \boldsymbol{m} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{b} = 0. \tag{2.26}$$

If the edge component of $\boldsymbol{b}$ does not vanish, that is, $b_{(m)} \neq 0$ in (2.26), then we can assume, without loss of generality, that

$$\rho b_{(m)} = \boldsymbol{m}\gamma\boldsymbol{l} + \mu > 0, \tag{2.27}$$

and the ordered triple $(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n})$ is uniquely determined up to its orientation. Thus, we have defined a congruence of mixed dislocations endowed with the family $\pi_{\boldsymbol{n}}(\boldsymbol{l}, \boldsymbol{m})$ of oriented local slip planes containing the local Burgers vector of the congruence everywhere. The two-dimensional distribution $\pi_{\boldsymbol{n}}(\boldsymbol{l}, \boldsymbol{m})$ is univocally defined by the edge component of the congruence of mixed dislocations. In particular, if the unit vector field $\boldsymbol{l}$ of (2.22) is an eigenvector of the symmetric tensor field γ, that is,

$$\gamma^{ab} l_b = \eta l^a, \quad l_a l^a = 1, \tag{2.28}$$

then

$$\rho\boldsymbol{b} = \eta\boldsymbol{l} + \mu\boldsymbol{m}, \quad \boldsymbol{l} \cdot \boldsymbol{m} = 0, \tag{2.29}$$

and

$$\rho b_g = (\eta^2 + \mu^2)^{1/2} > 0. \tag{2.30}$$

The corresponding congruence of mixed dislocations as well as the local glide system $(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n})$ of its edge component are called *principal* (Trzęsowski, 2000).

The eigenvector $\boldsymbol{l}$ of γ with the vanishing eigenvalue [$\eta = 0$ in (2.29)] defines a *principal congruence of edge dislocations*. In this case,

$$\rho\boldsymbol{b} = \mu\boldsymbol{m}, \quad \boldsymbol{l} \cdot \boldsymbol{m} = 0, \tag{2.31}$$

and, according to (2.20)–(2.22) and (2.30), we have

$$\rho b_g = (t_g/2)\sin\varphi, \quad 0 < \varphi < \pi. \tag{2.32}$$

If $\boldsymbol{\gamma} = \mathbf{0}$, then all lines, except those tangent to the direction of $\boldsymbol{t}$ for which $b_g = 0$, are edge dislocation lines.

3. INSTANTANEOUS CRYSTAL SURFACES

The Bravais moving frame can be time dependent: $\Phi = \Phi(t) = (\boldsymbol{E}_a(\cdot, t))$, $t \in I \subset R_+$, $[t] = s$. The object of anholonomity of (2.3) and the scalar volume dislocation density ρ depend then on the time as a parameter. The instantaneous intrinsic metric tensors $\boldsymbol{g}_t$, $t \in I$, are defined by

$$\begin{aligned} \boldsymbol{g}_t(X) = \boldsymbol{g}(X, t) &= \delta_{ab} E^a(X, t) \otimes E^b(X, t) \\ &= g_{AB}(X, t)\, dX^A \otimes dX^B, \\ g_{AB}(X, t) &= \overset{a}{e}_A(X, t)\overset{b}{e}_B(X, t)\delta_{ab}, \end{aligned} \tag{3.1}$$

where $\Phi^*(t) = (E^a(\cdot, t))$ is the moving coframe dual to $\Phi(t)$. The Riemannian material space $\mathcal{B}_g = (\mathcal{B}, \boldsymbol{g})$ now denotes a time-dependent material space and $\mathcal{B}_t = (\mathcal{B}, \boldsymbol{g}_t)$, $t \in I$, are the Riemannian *instantaneous material spaces*. All previous formulas describing a continuous distribution of dislocations or congruences of dislocations depend on the time as a parameter. In particular, we will deal, for a time-dependent local glide system, with the instantaneous local slip planes and slip surfaces.

Later, we assume additionally that the Bravais moving frame fulfils the following conditions:

$$[\boldsymbol{E}_\alpha, \boldsymbol{E}_\beta] = C^\kappa_{\alpha\beta} \boldsymbol{E}_\kappa, \quad \alpha, \beta, \kappa = 1, 2, \tag{3.2}$$

and

$$[\boldsymbol{E}_3, \boldsymbol{E}_\alpha] = H\boldsymbol{E}_\alpha, \quad \alpha = 1, 2, \tag{3.3}$$

where $C^\kappa_{\alpha\beta}$ and H are scalars defined on $\mathcal{B} \times I$. The condition (3.2) means that the two-dimensional distributions $\pi_t(\boldsymbol{E}_1, \boldsymbol{E}_2)$, $t \in I$, of instantaneous local crystal planes (Section 1) normal to the $\boldsymbol{E}_3$ direction are integrable (Sikorski, 1972; Von Westenholz, 1978) and their maximal integral manifolds are *instantaneous crystal surfaces*. These crystal surfaces are considered as two-dimensional submanifolds of the instantaneous material spaces (Section 2).

It is easy to see that if there exists a coordinate system $X = (X^A) = (X^\kappa, X^3)$, $[X^A] = \text{cm}$, such that for each $t \in I$ (Trzęsowski, 1997)

$$\boldsymbol{E}_\alpha(X, t) \overset{*}{=} \Psi_t^{-1/2}(X^3)\boldsymbol{a}_\alpha(X^\kappa, t), \qquad \boldsymbol{E}_3(X, t) \overset{*}{=} \partial_3, \tag{3.4}$$

where $\overset{*}{=}$ means that a relation is defined using a distinguished coordinate system, and

$$\Psi_t(X^3) = \lambda(t)^2 \exp[-2h_t(X^3)], \quad \Psi_t(0) = 1, \tag{3.5}$$

then the conditions (3.2) and (3.3) are satisfied with

$$C^{\kappa}_{\alpha\beta}(X, t) = \Psi_t^{-1}(X^3)c^{\kappa}_{\alpha\beta}(X^{\omega}, t), \quad [\boldsymbol{a}_{\alpha}, \boldsymbol{a}_{\beta}] = c^{\kappa}_{\alpha\beta}\boldsymbol{a}_{\kappa}, \tag{3.6}$$

and

$$H = H(X^3, t) = \partial_3 h_t(X^3). \tag{3.7}$$

Moreover, in this case, for each point $p \in \mathcal{B}$, the surfaces Σ_c, $c \in R$, defined as

$$\Sigma_c = \{q \in U\colon X^3(q) = c\}, \tag{3.8}$$

where $p \in U$ and U is a coordinate neighborhood for coordinates of (3.4)–(3.8), are time-independent slices of the instantaneous crystal surfaces. The above-defined coordinates will be called *adapted*. For any such coordinates, $\partial_{\alpha} = \partial/\partial X^{\alpha}$, $\alpha = 1, 2$, is a local basis for each two-dimensional instantaneous distribution $\pi_t(\boldsymbol{E}_1, \boldsymbol{E}_2)$, $t \in I$.

The instantaneous intrinsic metric tensor $\boldsymbol{g}_t$ of (3.1) takes, in adapted coordinates, the cannonical form of a metric tensor of the so-called *equidistant Riemannian space* (Trzęsowski, 1997):

$$\begin{aligned} \boldsymbol{g}_t(X) &= \boldsymbol{a}(X, t) + dX^3 \otimes dX^3, \\ \boldsymbol{a}(X, t) &= \Psi_t(X^3)\boldsymbol{a}_t(X^{\kappa}) = g_{\alpha\beta}(X, t)\, dX^{\alpha} \otimes dX^{\beta}, \end{aligned} \tag{3.9}$$

where

$$\boldsymbol{a}_t(X^{\kappa}) = \delta_{\alpha\beta}a^{\alpha}(X^{\kappa}, t) \otimes a^{\beta}(X^{\kappa}, t) = a_{\alpha\beta}(X^{\kappa}, t)\, dX^{\alpha} \otimes dX^{\beta} \quad \langle a^{\alpha}, \boldsymbol{a}_{\beta}, \rangle = \delta^{\alpha}_{\beta}, \tag{3.10}$$

and $\boldsymbol{a}_t$ is the metric tensor of a general two-dimensional Riemannian space. The treatment of instantaneous crystal surfaces as submanifolds of the instantaneous Riemannian material spaces $\mathcal{B}_t = (\mathcal{B}, \boldsymbol{g}_t)$ induces on the slices Σ_c, $c \in R$, of these surfaces a time-dependent geometric structure with the metric tensor $\boldsymbol{a}_{c,t}$ of the form

$$\boldsymbol{a}_{c,t}(X^{\kappa}) = \Psi_t(c)\boldsymbol{a}_t(X^{\kappa}). \tag{3.11}$$

It can be shown that the instantaneous crystal surfaces $\Sigma_{c,t} = (\Sigma_c, \boldsymbol{a}_{c,t})$, are umbilical with the constant mean curvature $H_t(c)$ given by (Trzęsowski, 1997)

$$H_t(c) = H(c, t), \tag{3.12}$$

where the definition of the mean curvature according to Schouten (1954), in place of the definition of Eisenhart (1964) that has been used in Trzęsowski (1997), was taken into account.

It is known that for any point of the two-dimensional (analytical) Riemannian manifold $\Sigma_{c,t}$ there is a neighborhood that has an (analytical) isometric embedding in the Euclidean configurational point space E^3 of the body (Friedman, 1965). The image of $\Sigma_{c,t}$ under this local embedding is a time-dependent surface in E^3 endowed with the same time-dependent Gaussian curvature $K_t(X^\kappa, c)$ as the submanifold $\Sigma_{c,t} \subset \mathcal{B}_t$ has. This quantity is obtainable by means of measurements within the surface; that is, it is an intrinsic geometric property of crystal surfaces due to the influence of secondary point defects. However, the mean curvature $H_t(c)$ of $\Sigma_{c,t}$ is not, in general, preserved under this embedding (since it is a relative geometric quantity). Consequently, the mean curvature has the physical meaning of a material parameter being, according to (3.3), a measure of the influence of secondary point defects on the long-range distortion of the continuously dislocated Bravais crystal. In particular, for a distribution of edge dislocations defined by the condition $\boldsymbol{\gamma} = \mathbf{0}$ (see remarks at the very end of Section 2) and by (3.2)–(3.4), the instantaneous crystal surfaces $\Sigma_{c,t}$ are flat manifolds, and thus this is virtually a *single glide* case (Section 1). Moreover, in this case (Trzęsowski, 2000)

$$[\boldsymbol{E}_1, \boldsymbol{E}_2] = \mathbf{0}, \quad [\boldsymbol{E}_3, \boldsymbol{E}_\alpha] = H\boldsymbol{E}_\alpha, \quad \alpha = 1, 2, \tag{3.13}$$

and thus the mean curvature $H(c, t)$ of the umbilical crystal surfaces $\Sigma_{c,t}$ is the only parameter describing the long-range distortion. Note that the instantaneous Riemannian material space $\mathcal{B}_t$ can be locally isometrically embedded in E^3 iff $\mathcal{B}_t$ is a flat Riemannian space [see the commentary following (2.4)]. For example, the Bravais moving frame $\Phi = (\boldsymbol{E}_a)$ such that

$$\begin{gathered}[\boldsymbol{E}_1, \boldsymbol{E}_2] = \gamma\boldsymbol{E}_3, \quad [\boldsymbol{E}_1, \boldsymbol{E}_3] = -\gamma\boldsymbol{E}_2, \quad [\boldsymbol{E}_2, \boldsymbol{E}_3] = \mathbf{0},\\ \gamma = \text{const} > 0, \quad [\gamma] = \text{cm}^{-1},\end{gathered} \tag{3.14}$$

describes a distribution of dislocations for which its Riemannian material space $\mathcal{B}_g$ is flat (Trzęsowski, 2000).

Let us denote by $\nabla^g = (\Gamma^A_{BC}[\boldsymbol{g}])$ the Levi-Civita covariant derivative based on the Christoffel symbols $\Gamma^A_{BC}[\boldsymbol{g}]$ corresponding to the metric tensor $\boldsymbol{g}$ defined by (3.1) and (3.9) and dependent on the time as a parameter. The Christoffel symbols have the form $\Gamma^A_{BC}[\boldsymbol{g}] = \Gamma^A_{BC}[\boldsymbol{g}_t]$, $t \in I$, where, according to (3.11) and (3.12) [Trzęsowski, 1997; see the remark following (3.12)]

$$\begin{aligned}&\Gamma^3_{33}[\boldsymbol{g}_t] = \Gamma^3_{3\alpha}[\boldsymbol{g}_t] = \Gamma^\alpha_{33}[\boldsymbol{g}_t] = 0,\\ &\Gamma^\alpha_{\beta 3} = -H_t\delta^\alpha_\beta, \quad \Gamma^3_{\alpha\beta}[\boldsymbol{g}_t] = H_t g_{\alpha\beta},\\ &\Gamma^\kappa_{\alpha\beta}[\boldsymbol{g}_t] = \Gamma^\kappa_{\alpha\beta}[\boldsymbol{a}_{c,t}] = \Gamma^\kappa_{\alpha\beta}[\boldsymbol{a}_t].\end{aligned} \tag{3.15}$$

Let $\nabla^a = (\Gamma^\kappa_{\alpha\beta}[\boldsymbol{a}_t])$ denote the Levi-Civita covariant derivative based on the Christoffel symbols $\Gamma^\kappa_{\alpha\beta}[\boldsymbol{a}_t]$, $t \in I$, corresponding to the metric tensor $\boldsymbol{a}_t$ of (3.10). If $\boldsymbol{u} = u^A \partial_A$ is a time-dependent vector field tangent to $\mathcal{B}_g = (\mathcal{B}, \boldsymbol{g})$, then the components $\nabla^g_A u^B$ of its covariant derivative $\nabla^g \boldsymbol{u}$ take, according to (3.15), the following form:

$$\nabla^g_\beta u^\alpha = \nabla^a_\beta u^\alpha - \delta^\alpha_\beta H u^3, \quad \nabla^g_\beta u^3 = \partial_\beta u^3 + H g_{\beta\kappa} u^\kappa,$$
$$\nabla^g_3 u^\alpha = \partial_3 u^\alpha - H u^\alpha, \quad \nabla^g_3 u^3 = \partial_3 u^3; \quad \alpha, \beta, \kappa = 1, 2, \tag{3.16}$$

where

$$\nabla^\alpha_\beta u^\alpha = \partial_\beta u^\alpha + \Gamma^\alpha_{\beta\kappa}[\boldsymbol{a}_t] u^\kappa. \tag{3.17}$$

In particular, if

$$\boldsymbol{u} \cdot \boldsymbol{E}_3 = 0, \quad \text{i.e., } \boldsymbol{u} = u^\alpha \partial_\alpha, \quad u^3 = 0, \tag{3.18}$$

then

$$\nabla^g_\beta u^\alpha = \nabla^a_\beta u^\alpha, \quad \nabla^g_\beta u^3 = H g_{\beta\kappa} u^\kappa,$$
$$\nabla^g_3 u^\alpha = \partial_3 u^\alpha - H u^\alpha, \quad \nabla^g_3 u^3 = 0, \tag{3.19}$$

and if $\boldsymbol{l} = l^A \partial_A$ is also orthogonal to $\boldsymbol{E}_3$, then the covariant derivative $\nabla^g_{\boldsymbol{l}} \boldsymbol{u}$ of $\boldsymbol{u}$ in the direction of $\boldsymbol{l}$ is given by

$$\nabla^g_{\boldsymbol{l}} \boldsymbol{u} = l^A \nabla^g_A \boldsymbol{u} = \nabla^a_{\boldsymbol{l}} \boldsymbol{u} + (H \boldsymbol{l} \cdot \boldsymbol{u}) \boldsymbol{E}_3,$$
$$\boldsymbol{E}_3 \cdot \nabla^a_{\boldsymbol{l}} \boldsymbol{u} = 0, \quad \boldsymbol{l} \cdot \boldsymbol{u} = g_{AB} l^A u^B = g_{\alpha\beta} l^\alpha u^\beta, \quad u^3 = l^3 = 0. \tag{3.20}$$

Moreover, it follows from (3.4) and (3.19) that

$$\nabla^g_{\boldsymbol{E}_3} \boldsymbol{u} = \partial_3 \boldsymbol{u} - H \boldsymbol{u}, \quad \boldsymbol{u} \cdot \boldsymbol{E}_3 = 0, \tag{3.21}$$

and, according to (3.4), (3.5), (3.7), and (3.21), we obtain

$$\nabla^g_{\boldsymbol{E}_3} \boldsymbol{E}_a = 0, \quad a = 1, 2, 3, \tag{3.22}$$

which means that the considered Bravais moving frame is ∇^g parallel along lattice lines (Section 1) normal to the family $\Sigma = \{\Sigma_{c,t}; c \in R, t \in I\}$ of instantaneous umbilical crystal surfaces and constituting a time-independent geodesic congruence in the time-dependent Riemannian material space $\mathcal{B}_g = (\mathcal{B}, \boldsymbol{g})$.

4. PRINCIPAL CONGRUENCES OF DISLOCATIONS

The notion of principal congruences of dislocations (Section 2) affords possibilities for the description of long-range distortions of continuously dislocated Bravais crystals in terms of these congruences. Consequently, any congruence of dislocations can be described in these terms.

If the Bravais moving frame is defined by (3.2) and (3.3), then it follows from (2.16) that

$$t_1 = C^2_{12}, \quad t_2 = C^1_{21}, \quad t_3 = 2H, \tag{4.1}$$

and the components α^{ab} of the dislocation density tensor $\boldsymbol{\alpha}$ of (2.6) constitute the following matrix:

$$\left(\alpha^{ab}; \begin{matrix} a \downarrow & 1,2,3 \\ b \rightarrow & 1,2,3 \end{matrix}\right) = \begin{pmatrix} 0 & t_3/2 & 0 \\ -t_3/2 & 0 & 0 \\ t_2 & -t_1 & 0 \end{pmatrix}. \tag{4.2}$$

The matrix of components γ^{ab} of the symmetric part $\boldsymbol{\gamma}$ of the dislocation density tensor has the form

$$\left(\gamma^{ab}; \begin{matrix} a \downarrow & 1,2,3 \\ b \rightarrow & 1,2,3 \end{matrix}\right) = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ \alpha & \beta & 0 \end{pmatrix}, \tag{4.3}$$

where

$$\alpha = t_2/2, \quad \beta = -t_1/2. \tag{4.4}$$

The eigenvectors $\boldsymbol{\gamma}_a$, $a = 1, 2, 3$, of the symmetric tensor $\boldsymbol{\gamma}$ of (2.18), computed with respect to the intrinsic metric tensor $\boldsymbol{g}$ of (3.1), are defined by

$$\boldsymbol{\gamma\gamma}_a = \gamma_a \boldsymbol{\gamma}_a, \quad \boldsymbol{\gamma}_a \cdot \boldsymbol{\gamma}_b = \delta_{ab}, \quad a, b = 1, 2, 3, \tag{4.5}$$

where the eigenvalues γ_a, $a = 1, 2, 3$, are roots of the determinant equation

$$\det(\gamma^{ab} - \lambda\delta^{ab}) = \lambda(\lambda - \gamma)(\lambda + \gamma) = 0, \quad \gamma = (\alpha^2 + \beta^2)^{1/2} \geq 0. \tag{4.6}$$

Introducing the angle θ by

$$\theta = \frac{1}{\sqrt{2}} \operatorname{arctg}\left(-\frac{\alpha}{\beta}\right), \tag{4.7}$$

and taking into account (4.4), we obtain

$$t_1 = -2\gamma \cos\sqrt{2}\theta, \quad t_2 = -2\gamma \sin\sqrt{2}\theta. \tag{4.8}$$

A straightforward computation shows that (4.5) is satisfied by the following eigenvectors of $\boldsymbol{\gamma}$:

$$\begin{aligned} \boldsymbol{\gamma}_1 &= \frac{1}{\sqrt{2}}(\boldsymbol{k} + \boldsymbol{E}_3), \quad \boldsymbol{\gamma}_2 = \frac{1}{\sqrt{2}}(\boldsymbol{k} - \boldsymbol{E}_3), \\ \boldsymbol{\gamma}_3 &= \cos\sqrt{2}\theta\, \boldsymbol{E}_1 + \sin\sqrt{2}\theta\, \boldsymbol{E}_2, \\ \boldsymbol{k} &= \sin\sqrt{2}\theta\, \boldsymbol{E}_1 - \cos\sqrt{2}\theta\, \boldsymbol{E}_2, \end{aligned} \tag{4.9}$$

with the corresponding eigenvalues of $\boldsymbol{\gamma}$ given by

$$-\gamma_1 = \gamma_2 = \gamma, \quad \gamma_3 = 0. \tag{4.10}$$

Thus, we obtain

$$\boldsymbol{\gamma} = \gamma(-\boldsymbol{\gamma}_1 \otimes \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2 \otimes \boldsymbol{\gamma}_2), \quad \gamma \geq 0, \tag{4.11}$$

and, according to (4.1), (4.8), and (4.9), the vector field $\boldsymbol{t}$ of (2.18) and (2.19) takes the form

$$\boldsymbol{t} = 2(-\gamma\boldsymbol{\gamma}_3 + H\boldsymbol{E}_3), \quad t_g = 2(\gamma^2 + H^2)^{1/2} > 0. \tag{4.12}$$

It follows from (4.9), (4.11), and (4.12) that [see (2.24)]

$$\boldsymbol{\gamma t} = -2\gamma H\boldsymbol{k}, \quad \gamma^2 + H^2 \neq 0. \tag{4.13}$$

Let us consider a general congruence of dislocations defined by (2.19)–(2.22), (4.9), (4.11), and (4.12). The local Burgers vector $\boldsymbol{b}$ of the congruence is given by

$$\begin{gathered}\rho\boldsymbol{b} = \gamma[-(\boldsymbol{l}\cdot\boldsymbol{\gamma}_1)\boldsymbol{\gamma}_1 + (\boldsymbol{l}\cdot\boldsymbol{\gamma}_2)\boldsymbol{\gamma}_2] + (t_g/2)\boldsymbol{M},\\ (t_g/2)\boldsymbol{M} = \gamma\boldsymbol{K} + H[(\boldsymbol{l}\cdot\boldsymbol{E}_2)\boldsymbol{E}_1 - (\boldsymbol{l}\cdot\boldsymbol{E}_1)\boldsymbol{E}_2], \quad \boldsymbol{M}\cdot\boldsymbol{l} = 0,\\ \boldsymbol{K} = \cos\varphi_{\boldsymbol{l},\boldsymbol{E}_3}\boldsymbol{k} - \cos\varphi_{\boldsymbol{l},\boldsymbol{k}}\boldsymbol{E}_3, \quad \cos\varphi_{\boldsymbol{a},\boldsymbol{b}} = \boldsymbol{a}\cdot\boldsymbol{b}/\|\boldsymbol{a}\|_g\|\boldsymbol{b}\|_g,\end{gathered} \tag{4.14}$$

and

$$\cos\varphi_{\boldsymbol{l},\boldsymbol{b}} = (2\gamma/\rho b_g)\cos\varphi_{\boldsymbol{l},\boldsymbol{k}}\cos\varphi_{\boldsymbol{l},\boldsymbol{E}_3}. \tag{4.15}$$

It follows that the congruence consists of edge dislocations iff

$$\gamma\cos\varphi_{\boldsymbol{l},\boldsymbol{k}}\cos\varphi_{\boldsymbol{l},\boldsymbol{E}_3} = 0, \tag{4.16}$$

or it consists of screw dislocations iff

$$\rho b_g = 2\gamma|\cos\varphi_{\boldsymbol{l},\boldsymbol{k}}\cos\varphi_{\boldsymbol{l},\boldsymbol{E}_3}| > 0. \tag{4.17}$$

If the Bravais moving frame is defined by (3.4)–(3.7), then

$$\begin{gathered}t_1(X,t) = \Psi_t^{-1/2}(X^3)c_{12}^2(X^\kappa,t),\\ t_2(X,t) = \Psi_t^{-1/2}(X^3)c_{21}^1(X^\kappa,t), \quad t_3(X,t) = 2H(X^3,t)\end{gathered} \tag{4.18}$$

and thus, in the adapted coordinates (Section 3), we have (Trzęsowski, 2000)

$$\partial_3\theta = 0, \tag{4.19}$$

where (4.4) and (4.7) were taken into account. In this case, the scalar H defines the constant mean curvature $H_t(c)$ [see (3.12)] of instantaneous umbilical crystal surfaces $\Sigma_{c,t}$, $c \in R$, $t \in I$, normal to the $\boldsymbol{E}_3$ direction (Section 3), and the condition

$$H = 0, \tag{4.20}$$

means that these surfaces are minimal varieties (Eisenhart, 1964). It follows from (4.12)–(4.14) that

$$\gamma \boldsymbol{t} = 0, \quad \boldsymbol{t} = -t_g \gamma_3, \quad t_g = 2\gamma > 0, \tag{4.21}$$

which means, according to (2.20)–(2.22), that the eigenvector γ_3 does not define a congruence of dislocation lines. Therefore, if the condition (4.20) is fulfilled, then the local Burgers vector $\boldsymbol{b}$ of a congruence of dislocations is given by

$$\rho \boldsymbol{b} = -2\gamma \cos \varphi_{\boldsymbol{l},\boldsymbol{k}} \boldsymbol{E}_3, \quad \boldsymbol{l} \neq \pm\gamma_3, \tag{4.22}$$

and

$$\rho b_g = 2\gamma |\cos \varphi_{\boldsymbol{l},\boldsymbol{k}}| > 0, \tag{4.23}$$

where (4.9), (4.14), (4.21), and (4.22) were taken into account.

5. EDGE DISLOCATIONS

A comparison of (4.16) with (4.20)–(4.23) leads to the conclusion that a class of congruences of edge dislocations that permits us to consider the particular case of crystal surfaces being minimal varieties is defined by the following condition:

$$\cos \varphi_{\boldsymbol{l},\boldsymbol{E}_3} = \boldsymbol{l} \cdot \boldsymbol{E}_3 = 0. \tag{5.1}$$

It follows from (4.14) that

$$\rho \boldsymbol{b} = H[(\boldsymbol{l} \cdot \boldsymbol{E}_2)\boldsymbol{E}_1 - (\boldsymbol{l} \cdot \boldsymbol{E}_1)\boldsymbol{E}_2] - 2\gamma \cos \varphi_{\boldsymbol{l},\boldsymbol{k}} \boldsymbol{E}_3, \tag{5.2}$$

and

$$\rho \boldsymbol{b}_g = (H^2 + 4\gamma^2 \cos^2 \varphi_{\boldsymbol{l},\boldsymbol{k}})^{1/2} > 0. \tag{5.3}$$

For example, it is the case of *principal glide system* ($\boldsymbol{l}$, $\boldsymbol{m}$, $\boldsymbol{n}$) defined by [see Section 2 and (4.9) and (4.10)]

$$\boldsymbol{l} = \gamma_3, \quad \boldsymbol{m} = \frac{1}{\sqrt{2}}(\gamma_1 + \gamma_2) = \boldsymbol{k}, \quad \boldsymbol{n} = \frac{1}{\sqrt{2}}(\gamma_1 - \gamma_2) = \boldsymbol{E}_3, \tag{5.4}$$

which defines, according to (5.1)–(5.3), a *principal congruence of edge dislocations* such that

$$\rho \boldsymbol{b} = H\boldsymbol{k}, \quad H > 0, \tag{5.5}$$

and

$$\rho b_g = H. \tag{5.6}$$

In this case,

$$\boldsymbol{l} \cdot \boldsymbol{E}_3 = \boldsymbol{l} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{E}_3 = 0. \tag{5.7}$$

If the Bravais moving frame is defined by (3.4)–(3.7), then the normal curvature κ_n of the instantaneous umbilical crystal surfaces normal to the $\boldsymbol{E}_3$ direction (Section 3) is the same for all their tangent directions and (Eisenhart, 1964)

$$\kappa_n = H. \tag{5.8}$$

It follows from (5.6) and (5.8) that the following counterpart of the approximate formula (1.1) holds (Trzęsowski, 2000):

$$\kappa_n = \rho b_g. \tag{5.9}$$

The curvature vector $\boldsymbol{\kappa}$ of the congruence (Eisenhart, 1964) can be written, according to (3.20) and (5.7), in the form

$$\boldsymbol{\kappa} = \nabla^g_{\boldsymbol{l}} \boldsymbol{l} = \boldsymbol{\kappa}_r + \boldsymbol{\kappa}_n, \quad \boldsymbol{\kappa}_r \cdot \boldsymbol{\kappa}_n = 0, \tag{5.10}$$

where

$$\boldsymbol{\kappa}_r = \nabla^a_{\boldsymbol{l}} \boldsymbol{l} = \kappa_r \boldsymbol{m}_r, \quad \boldsymbol{\kappa}_n = \kappa_n \boldsymbol{n}, \tag{5.11}$$

and (5.4) and (5.9) were taken into account. If for each $(X^3, t) \in R \times I$ [see (3.4)]

$$\nabla^a_{\boldsymbol{l}} \boldsymbol{E}_\alpha = 0, \quad \alpha = 1, 2, \tag{5.12}$$

then, according to (4.9), (5.4), and (5.11), the following generalized formulas of Frenet for a 2-manifold (cf. Hicks, 1965) hold:

$$\nabla^a_{\boldsymbol{l}} \boldsymbol{l} = \kappa_r \boldsymbol{m}_r, \quad \nabla^a_{\boldsymbol{l}} \boldsymbol{m}_r = -\kappa_r \boldsymbol{l}, \tag{5.13}$$

where

$$\boldsymbol{m}_r = -\boldsymbol{k}, \quad \kappa_r = \sqrt{2}\partial_{\boldsymbol{l}}\theta > 0. \tag{5.14}$$

So, the curvature κ of the congruence has the form

$$\kappa = \left(\kappa_r^2 + \kappa_n^2\right)^{1/2} = [2(\partial_{\boldsymbol{l}}\theta)^2 + H^2]^{1/2}. \tag{5.15}$$

and the formulas (5.5), (5.7), and (5.13) mean that edge dislocation lines of the congruence lie on the instantaneous umbilical crystal surfaces normal to the $\boldsymbol{E}_3$ direction and can glide in these surfaces. Thus, the instantaneous crystal surfaces are virtually *glide surfaces* (Section 1) for the principal congruence of edge dislocations. The time-dependent scalars $\kappa_r = \kappa_r(X^\kappa, X^3, t)$ and $\kappa_n = \kappa_n(X^3, t)$ are, for $X^3 = c$ and at each instant $t \in I$, the relative curvature of the congruence restricted to the crystal surface $\Sigma_{c,t} = (\Sigma_c, \boldsymbol{a}_{c,t})$ and the normal curvature of this surface for the $\boldsymbol{l}$ direction, respectively.

If the crystal surfaces normal to the $\boldsymbol{E}_3$ direction are *minimal varieties*, that is, the condition (4.20) is satisfied, then the case

$$\boldsymbol{l} = \boldsymbol{k}, \quad \text{i.e., } \cos\varphi_{\boldsymbol{l},\boldsymbol{k}} = \mathbf{1}, \tag{5.16}$$

defines, according to (5.1) and (5.2), a congruence of edge dislocations such that

$$\rho \boldsymbol{b} = -2\gamma \boldsymbol{E}_3, \quad \gamma > 0, \tag{5.17}$$

and

$$\rho b_g = 2\gamma. \tag{5.18}$$

So, comparing (5.16) and (5.17) with (5.4), we can define the corresponding local glide system of the congruence by

$$(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}) = (\boldsymbol{k}, -\boldsymbol{E}_3, \gamma_3). \tag{5.19}$$

If, additionally, the condition (5.12) with $\boldsymbol{l} = \boldsymbol{k}$ is satisfied, then

$$\begin{aligned} \boldsymbol{\kappa} = \nabla^g_{\boldsymbol{l}} \boldsymbol{l} = \nabla^a_{\boldsymbol{l}} \boldsymbol{l} = -\kappa \gamma_3, \\ \nabla^g_{\boldsymbol{l}} \gamma_3 = \nabla^a_{\boldsymbol{l}} \gamma_3 = \kappa \boldsymbol{l}, \\ \nabla^g_{\boldsymbol{l}} \boldsymbol{E}_3 = \boldsymbol{0}, \quad \kappa = \sqrt{2}\partial_{\boldsymbol{l}} \theta, \end{aligned} \tag{5.20}$$

where (3.20), (4.9), (4.20), and (5.16) were taken into account. We conclude, taking the curvature κ and the torsion τ of the congruence in the form

$$\kappa = \sqrt{2}\partial_{\boldsymbol{l}} \theta > 0, \quad \tau = 0, \tag{5.21}$$

that the formulas (5.20) are the Frenet generalized formulas for a congruence in the Riemannian space $\mathcal{B}_g = (\mathcal{B}, \boldsymbol{g})$ endowed with the Frenet vectors $(\boldsymbol{e}_a; a = 1, 2, 3)$ (cf. Hicks, 1965) of the form

$$\boldsymbol{e}_1 = \boldsymbol{k}, \quad \boldsymbol{e}_2 = -\gamma_3, \quad \boldsymbol{e}_3 = -\boldsymbol{E}_3. \tag{5.22}$$

So, we have defined a congruence of edge dislocation lines of zero torsion located on the instantaneous crystal surfaces normal to the local Burgers vector direction [see (5.17)]. Moreover, it follows from (3.3), (3.4), (3.7), (4.9), (4.19), (4.20), and (5.16) that we have

$$[\boldsymbol{E}_3, \boldsymbol{k}] = 0. \tag{5.23}$$

This means, according to (5.16) and (5.17), that the congruence consists of prismatic edge dislocations (Section 1) with *slip surfaces* (Sections 1 and 2) normal to the crystal surfaces.

6. HELICAL DISLOCATIONS

Let us consider a congruence of dislocation lines (edge or mixed) with the local Burgers vector $\boldsymbol{b}$ defined by (2.19)–(2.22). The formulas (5.20)–(5.22) suggest that we consider an orthonormal one-parameter base $(\boldsymbol{e}_a(\cdot, t); a = 1, 2, 3)$ of vector fields on $\mathcal{B}_t$, $t \in I$, such that $\boldsymbol{e}_1 = \boldsymbol{l}$ and, for each instant $t \in I$, the following *generalized formulas of Frenet* are valid:

$$\begin{aligned} \boldsymbol{\kappa} = \nabla^g_{\boldsymbol{l}} \boldsymbol{l} &= \kappa \boldsymbol{e}_2, \quad \kappa > 0, \\ \nabla^g_{\boldsymbol{l}} \boldsymbol{e}_2 &= -\kappa \boldsymbol{l} + \tau \boldsymbol{e}_3, \\ \nabla^g_{\boldsymbol{l}} \boldsymbol{e}_3 &= -\tau \boldsymbol{e}_2, \quad \tau \geq 0. \end{aligned} \tag{6.1}$$

The base $(\boldsymbol{e}_a)$ consists then of *Frenet vectors* of the congruence: $\boldsymbol{e}_l = \boldsymbol{l}$ is the (instantaneous) *tangent*, $\boldsymbol{e}_2$ is the (instantaneous) *principal normal*, and $\boldsymbol{e}_3$ is the (instantaneous) *second normal* of the time-dependent congruence of dislocations. The vector field $\boldsymbol{\kappa}$ is the (instantaneous) *curvature vector* of the congruence. The scalars κ and τ denote the (instantaneous) *curvature* and *torsion* of the congruence, respectively.

Let us define, as an example, the Frenet vectors for a congruence of helical dislocations (Section 1) consisting of cylindrical helices, as defined by the condition that the curvature and torsion of the congruence maintain a constant ratio (Laugwitz, 1965):

$$\tau = c\kappa, \quad c = \text{const} \geq 0, \tag{6.2}$$

where, in general, the dimensionless constant c can be dependent on the time parameter. The congruence of such curves is defined by the following generalized formulas of Frenet:

$$\nabla^g_{\boldsymbol{l}} \boldsymbol{l} = \kappa \boldsymbol{e}_2, \quad \kappa > 0, \tag{6.3a}$$

$$\nabla^g_{\boldsymbol{l}} \boldsymbol{e}_2 = -\kappa \boldsymbol{l} + c\kappa \boldsymbol{e}_3, \tag{6.3b}$$

$$\nabla^g_{\boldsymbol{l}} \boldsymbol{e}_3 = -c\kappa \boldsymbol{e}_2, \quad c \geq 0. \tag{6.3c}$$

It follows from (6.3a) and (6.3c) that

$$\begin{aligned} c\boldsymbol{l} + \boldsymbol{e}_3 &= \boldsymbol{a}, \quad \nabla^g_{\boldsymbol{l}} \boldsymbol{a} = 0, \\ a^2 &= \|\boldsymbol{a}\|^2_g = 1 + c^2 = \text{const.}, \end{aligned} \tag{6.4}$$

where the vector field $\boldsymbol{a}$ as well as its modulus $a > 0$ can be dependent on the time parameter, and the unit tangent $\boldsymbol{l}$ is inclined, at each instant $t \in I$, at the constant angle $\varphi_{\boldsymbol{l},\boldsymbol{a}}$ to the vector field $\boldsymbol{a}$:

$$\cos \varphi_{\boldsymbol{l},\boldsymbol{a}} = \frac{\boldsymbol{l} \cdot \boldsymbol{a}}{a} = \frac{c}{a}, \quad 0 \leq \varphi_{\boldsymbol{l},\boldsymbol{a}} < \pi/2 \tag{6.5}$$

Differentiating covariantly (6.3b) in the direction of $\boldsymbol{l}$, substituting (6.3a), and taking into account (6.4), we obtain

$$\nabla^g_{\boldsymbol{l}} \nabla^g_{\boldsymbol{l}} \boldsymbol{e}_2 + a^2 \kappa^2 \boldsymbol{e}_2 = \partial_{\boldsymbol{l}} \kappa (c\boldsymbol{a} - a^2 \boldsymbol{l}). \tag{6.6}$$

It is easy to see that if

$$\boldsymbol{l} = \boldsymbol{e}_1 = \frac{1}{a}(\sin a\theta\,\boldsymbol{E}_1 - \cos a\theta\,\boldsymbol{E}_2 + c\boldsymbol{E}_3),$$
$$\boldsymbol{e}_2 = \cos a\theta\,\boldsymbol{E}_1 + \sin a\theta\,\boldsymbol{E}_2, \quad \boldsymbol{a} = a\boldsymbol{E}_3, \tag{6.7}$$

where $\Phi = (\boldsymbol{E}_a)$ is a Bravais moving frame such that

$$\nabla^g_{\boldsymbol{l}}\boldsymbol{E}_a = 0, \quad a = 1, 2, 3, \tag{6.8}$$

and the curvature κ of the congruence has the form

$$\kappa = \partial_{\boldsymbol{l}}\theta > 0, \tag{6.9}$$

then the conditions (6.4)–(6.6) are satisfied with

$$\boldsymbol{e}_3 = -\frac{c}{a}(\sin a\theta\,\boldsymbol{E}_1 - \cos a\theta\,\boldsymbol{E}_2) + \frac{1}{a}\boldsymbol{E}_3. \tag{6.10}$$

Let us consider a Bravais moving frame defined by the conditions (3.2) and (3.3), and let the angle θ of (6.7)–(6.10) cover that one of (4.7). Then, according to (4.9), (4.11), (4.14), and (6.7), the formula (2.22) holds, where

$$\gamma\boldsymbol{l} = -\frac{\gamma}{a}[c\boldsymbol{k} + \cos(a - \sqrt{2})\theta\,\boldsymbol{E}_3], \tag{6.11}$$

and

$$\mu\boldsymbol{m} = -\frac{\gamma c}{a}\boldsymbol{k} - \frac{1}{a}[H(\cos a\theta\,\boldsymbol{E}_1 + \sin a\theta\,\boldsymbol{E}_2) + \gamma\cos(a - \sqrt{2})\theta\,\boldsymbol{E}_3]. \tag{6.12}$$

Thus, the congruence consists of mixed helical dislocations and its local Burgers vector is given by

$$\rho\boldsymbol{b} = -\frac{1}{a}[H(\cos a\theta\,\boldsymbol{E}_1 + \sin a\theta\,\boldsymbol{E}_2) + \gamma(1 + c)\cos(a - \sqrt{2})\theta\,\boldsymbol{E}_3]. \tag{6.13}$$

If

$$c = 1, \quad \text{i.e., } a = \sqrt{2}, \tag{6.14}$$

then (6.11) and (6.12) reduce to

$$\gamma\boldsymbol{l} = -\gamma\boldsymbol{l}, \quad \boldsymbol{l} = \gamma_1, \tag{6.15}$$

and

$$\mu\boldsymbol{m} = \gamma\gamma_2 - \frac{H}{\sqrt{2}}\gamma_3. \tag{6.16}$$

The formula (6.13) then takes the following form:

$$\rho\boldsymbol{b} = -\frac{H}{\sqrt{2}}\gamma_3 - \sqrt{2}\gamma\,\boldsymbol{E}_3. \tag{6.17}$$

So, it is a *principal congruence of mixed helical dislocations*. Note that if the Bravais moving frame is defined by (3.4)–(3.7), then it follows from (3.16)–(3.19), (3.22), (4.9), (6.7), and (6.14) that the condition (6.8) is equivalent to the following conditions:

$$\nabla^a_{\boldsymbol{k}} \boldsymbol{E}_\alpha = 0, \quad \alpha = 1, 2, \tag{6.18}$$

and

$$H = 0. \tag{6.19}$$

The local Burgers vector $\boldsymbol{b}$ then takes the form

$$\rho \boldsymbol{b} = -\gamma \boldsymbol{a}, \quad \boldsymbol{a} = \sqrt{2}\boldsymbol{E}_3, \tag{6.20}$$

with

$$\rho b_g = \sqrt{2}\gamma, \tag{6.21}$$

and $\boldsymbol{b}$ is inclined, at each instant $t \in I$, at the constant angle $\varphi_{\boldsymbol{l},\boldsymbol{b}} = 3\pi/4$ to the unit tangent $\boldsymbol{l}$ of dislocation lines. The corresponding principal local glide system (Section 2) and the Frenet moving trihedron are given by

$$\begin{aligned} (\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}) &= (\gamma_1, \gamma_2, \gamma_3), \\ (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) &= (\gamma_1, \gamma_3, -\gamma_2). \end{aligned} \tag{6.22}$$

The formulas (6.2) and (6.9) reduce to

$$\tau = \kappa = \frac{1}{\sqrt{2}} \partial_{\boldsymbol{k}} \theta > 0, \tag{6.23}$$

where (4.19) and (6.14) are taken into account.

It follows from (4.9), (6.7), and (6.11) that

$$\boldsymbol{l}\gamma\boldsymbol{l} = -\frac{\gamma c}{a^2} \cos(a - \sqrt{2})\theta, \quad \gamma > 0, \tag{6.24}$$

and thus, according to (2.23), the considered congruence of cylindrical helices consists of edge dislocation lines if

$$c = 0, \quad \text{i.e.}, a = 1. \tag{6.25}$$

Then

$$\begin{aligned} \boldsymbol{l} = \boldsymbol{e}_1 &= \sin\theta \boldsymbol{E}_1 - \cos\theta \boldsymbol{E}_2, \\ \boldsymbol{e}_2 &= \cos\theta \boldsymbol{E}_1 + \sin\theta \boldsymbol{E}_2, \\ \boldsymbol{e}_3 = \boldsymbol{a} &= \boldsymbol{E}_3, \end{aligned} \tag{6.26}$$

and

$$\gamma \boldsymbol{l} = -\gamma \cos(\sqrt{2}-1)\theta \boldsymbol{E}_3,$$
$$\mu \boldsymbol{m} = -H\boldsymbol{e}_2 - \gamma \cos(\sqrt{2}-1)\theta \boldsymbol{E}_3. \tag{6.27}$$

In particular, if the instantaneous crystal surfaces are minimal varieties, that is, the condition (6.19) is fulfilled, then

$$\rho \boldsymbol{b} = -2\gamma \cos(\sqrt{2}-1)\theta \boldsymbol{E}_3, \tag{6.28}$$

with

$$\rho b_g = \mu > 0,$$
$$\mu = 2\gamma \cos(\sqrt{2}-1)\theta, \quad 0 \leq \theta < \pi/2(\sqrt{2}-1). \tag{6.29}$$

Thus, the corresponding local glide system has the form

$$(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}) = (\boldsymbol{e}_1, -\boldsymbol{e}_3, \boldsymbol{e}_2), \tag{6.30}$$

and the curvature and torsion of the congruence are given by

$$\kappa = \partial_{\boldsymbol{l}}\theta > 0, \quad \tau = 0. \tag{6.31}$$

If the Bravais moving frame is defined by (3.4)–(3.7), then the condition (6.18) with $\boldsymbol{k} = \boldsymbol{l}$ and the condition (6.19) are satisfied. Moreover, in this case, the formula (5.23) with $\boldsymbol{k} = \boldsymbol{l}$ is valid. So, we have defined a congruence of *helical prismatic edge dislocations* of torsion zero analogous to the one discussed at the very end of Section 5.

7. KINEMATICS OF CONGRUENCES OF DISLOCATIONS

Let us consider a congruence of mixed dislocations endowed with the local glide system $(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n})$ uniquely determined up to its orientation (Section 2). If $(\boldsymbol{e}_a; a = 1, 2, 3)$ is the Frenet moving trihedron of the congruence (Section 6), then

$$\boldsymbol{e}_1 = \boldsymbol{l}, \quad \boldsymbol{e}_2 = \cos\vartheta \boldsymbol{m} + \sin\vartheta \boldsymbol{n},$$
$$\boldsymbol{e}_3 = -\sin\vartheta \boldsymbol{m} + \cos\vartheta \boldsymbol{n}. \tag{7.1}$$

It follows from (2.26), (6.1), and (7.1) that

$$\nabla_{\boldsymbol{l}}^{g}\boldsymbol{b} = [\partial_{\boldsymbol{l}} b_{(l)} - b_{(m)}\kappa\cos\vartheta]\boldsymbol{l} + [\partial_{\boldsymbol{l}} b_{(m)} + b_{(l)}\kappa\cos\vartheta]\boldsymbol{m}$$
$$+ [b_{(m)}(\tau - \partial_{\boldsymbol{l}}\vartheta) + b_{(l)}\kappa\sin\vartheta]\boldsymbol{n}. \tag{7.2}$$

Therefore, at each body point, the local Burgers vector $\boldsymbol{b}$ of the congruence as well as its variation $\nabla_{\boldsymbol{l}}^{g}\boldsymbol{b}$ in the $\boldsymbol{l}$ direction are located in the same local slip plane

(Section 2) normal to the $\boldsymbol{n}$ direction iff

$$b_{(m)}(\tau - \partial_{\boldsymbol{l}}\vartheta) + b_{(l)}\kappa \sin\vartheta = 0, \quad \kappa > 0. \tag{7.3}$$

Note that, according to (2.11) and (2.26), the congruence consists of edge dislocations iff

$$b_{(l)} = \boldsymbol{b} \cdot \boldsymbol{l} = 0, \quad b_{(m)} \neq 0, \tag{7.4}$$

where (2.27) was taken into account. So, in this case, the condition (7.3) reduces to the following representation of the torsion τ of the congruence:

$$\tau = \partial_{\boldsymbol{l}}\vartheta \geq 0. \tag{7.5}$$

In the following, we will consider the congruences of mixed dislocations restricted by the above condition. This means that the *climb component* (Section 1)

$$\boldsymbol{n} \cdot \nabla_{\boldsymbol{l}}^{g}\boldsymbol{b} = b_{(l)}\kappa \sin\vartheta, \quad \boldsymbol{n} \cdot \boldsymbol{b} = 0, \tag{7.6}$$

of the local Burgers variation is admitted.

Equation (7.1) can be rewritten in the following complex form:

$$\boldsymbol{N} = \boldsymbol{m} + i\boldsymbol{n} = (\boldsymbol{e}_2 + i\boldsymbol{e}_3)e^{i\vartheta}, \quad \boldsymbol{l} = \boldsymbol{e}_1, \tag{7.7}$$

where

$$\boldsymbol{N} \cdot \boldsymbol{N} = \boldsymbol{l} \cdot \boldsymbol{N} = 0, \quad \boldsymbol{N} \cdot \boldsymbol{N}^* = 2, \quad \boldsymbol{l} \cdot \boldsymbol{l} = 1, \tag{7.8}$$

and the asterisk denotes the complex conjugation. Introducing the complex variable ψ of the form

$$\psi = \kappa e^{i\vartheta}, \quad \kappa > 0, \tag{7.9}$$

where κ is the curvature of the congruence, and taking into account the formula (7.5), we can rewrite the generalized formulas of Frenet (6.1) in terms of the local glide system $(\boldsymbol{l}, \boldsymbol{N})$ and the complex variale ψ:

$$\boldsymbol{\kappa} = \frac{1}{2}(\psi^*\boldsymbol{N} + \psi\boldsymbol{N}^*), \quad \nabla_{\boldsymbol{l}}^{g}\boldsymbol{N} = -\psi\boldsymbol{l}. \tag{7.10}$$

Note that the unknown time-dependent scalars κ and τ of the generalized Frenet formulas (6.1) can be treated as those that distinguish one class of congruences of moving dislocations from another [see (7.1) or (7.7)]. Consequently, the complex version (7.10) of these formulas needs additional *kinematic equations* defining the evolution of curvature and torsion of a congruence of moving dislocations. A method of deriving such equations, based on the Frenet formulas for a single curve in the Euclidean space R^3, has been formulated in order to describe the motion of a very thin isolated vortex filament (Hashimoto, 1972; see also Lamb, 1977, 1980). The method can be generalized in order to describe a congruence of time-dependent curves in a Riemannian space. Namely, by putting

$$\partial_t \boldsymbol{N} = \omega_1 \boldsymbol{N} + \omega_2 \boldsymbol{N}^* + \omega \boldsymbol{l},$$
$$\partial_t \boldsymbol{l} = \omega_3 \boldsymbol{N} + \omega_4 \boldsymbol{N}^* + \omega_5 \boldsymbol{l}, \tag{7.11}$$

and noting the relations of (7.8) and their partial derivatives with respect to time, we obtain

$$\omega_1 = i\zeta, \quad \omega_2 = \omega_5 = 0, \quad \omega_3 = -\omega^*/2, \quad \omega_4 = -\omega/2, \tag{7.12}$$

where ω and ζ denote the complex and real scalars defined on $\mathcal{B} \times I$, respectively. So, we have

$$\partial_t \boldsymbol{N} = \omega \boldsymbol{l} + i\zeta \boldsymbol{N},$$
$$\partial_t \boldsymbol{l} = -\frac{1}{2}(\omega^* \boldsymbol{N} + \omega \boldsymbol{N}^*), \quad [\omega] = [\zeta] = \mathrm{s}^{-1}. \tag{7.13}$$

The condition

$$\partial_t \nabla^g_{\boldsymbol{l}} \boldsymbol{N} = \nabla^g_{\boldsymbol{l}}(\partial_t \boldsymbol{N}), \tag{7.14}$$

puts the following constrains on (7.10) and (7.13):

$$\partial_t \psi + \partial_{\boldsymbol{l}} \omega - i\zeta\psi = 0, \tag{7.15a}$$

$$\partial_{\boldsymbol{l}} \zeta = \frac{i}{2}(\omega\psi^* - \omega^*\psi) = \mathrm{Im}(\omega^*\psi). \tag{7.15b}$$

Note that (7.15b) means that the following condition should be fulfilled:

$$\partial_t \boldsymbol{\kappa} = \nabla^g_{\boldsymbol{l}}(\partial_t \boldsymbol{l}). \tag{7.16}$$

The system of Eqs. (7.15) is not closed, and thus some additional conditions are needed. Let us assume, for example, that the scalar ω of (7.15) is real. It reduces (7.9) and (7.15) to the following system of three real equations for four real variables κ, ϑ, ζ, and ω:

$$\partial_t \kappa + \cos\vartheta\, \partial_{\boldsymbol{l}} \omega = 0,$$
$$\kappa(\zeta - \partial_t \vartheta) + \sin\vartheta\, \partial_{\boldsymbol{l}} \omega = 0,$$
$$\partial_{\boldsymbol{l}} \zeta = \omega\kappa \sin\vartheta, \tag{7.17}$$

where the versor $\boldsymbol{l}$ is treated as a fixed variable. Let us take, as an example, the principal congruence of edge dislocations defined by (3.2)–(3.7), (3.22), (4.19), (5.1), and (5.4)–(5.15). The principal normal $\boldsymbol{e}_2$ of the congruence (Section 6) can be written in the form

$$\boldsymbol{e}_2 = -\sin\sigma \boldsymbol{m} + \cos\sigma \boldsymbol{n},$$
$$\sin\sigma = \kappa_r/\kappa, \quad \cos\sigma = H/\kappa, \quad \kappa = \left(\kappa_r^2 + H^2\right)^{1/2}, \tag{7.18}$$

where (5.4), (5.8), and (5.10)–(5.15) were taken into account. Comparing (7.1) with (7.18), we obtain

$$\sigma = 3\pi/2 + \vartheta,$$
$$\sigma = \text{arctg}(\kappa_r/H), \quad \kappa_r = \sqrt{2}\partial_{\boldsymbol{l}}\theta, \tag{7.19}$$

and thus, taking into account (3.4), (4.9), (4.19), and (5.4), we obtain the following additional condition:

$$\kappa \sin\vartheta = H, \quad 0 < \vartheta < \pi, \quad \partial_{\boldsymbol{l}} H = 0, \tag{7.20}$$

where $H = H(X^3, t)$ is, for $X^3 = c$, the mean curvature of the instantaneous crystal surfaces $\Sigma_{c,t}$, $t \in I$ (Section 3) being virtually glide surfaces for dislocations of the congruence (Section 5). Note that it follows from (2.26), (5.4)–(5.6), (7.2)–(7.5), and (7.20) that the variation $\nabla^g_{\boldsymbol{l}}\boldsymbol{b}$ in the $\boldsymbol{l}$ direction of the local Burgers vector $\boldsymbol{b}$ of the congruence has the form

$$\nabla^g_{\boldsymbol{l}}\boldsymbol{b} = -\frac{H}{\rho}\left[Hctg\vartheta\boldsymbol{l} + \partial_{\boldsymbol{l}} ln\left(\frac{\rho}{\rho_0}\right)\boldsymbol{m}\right]. \tag{7.21}$$

Straightforward computations show that the system of Eqs. (7.17) and (7.20) leads to the following nonlinear evolution equation:

$$\partial_t\kappa + \frac{1}{H\kappa}\sqrt{\kappa^2 - H^2}\partial_{\boldsymbol{l}}\partial_{\boldsymbol{l}}\zeta = 0 \tag{7.22}$$

where

$$\zeta = \frac{\partial_t H}{\sqrt{\kappa^2 - H^2}}, \quad \partial_{\boldsymbol{l}} H = 0. \tag{7.23}$$

Moreover,

$$\vartheta = \arcsin(H/\kappa), \tag{7.24}$$

and

$$\omega = \frac{1}{H}\partial_{\boldsymbol{l}}\zeta + \omega_0, \quad \partial_{\boldsymbol{l}}\omega_0 = 0. \tag{7.25}$$

Note that if

$$\partial_t H = 0, \quad \text{i.e., } H = H(X^3) \tag{7.26}$$

then, according to (7.22)–(7.25), the system of Eqs. (7.17) reduces to

$$\zeta = \omega = 0, \quad \partial_t\kappa = 0, \quad \partial_t\vartheta = 0 \tag{7.27}$$

and it follows from (7.5), (7.7), and (7.13) that

$$\partial_t\boldsymbol{l} = \partial_t\boldsymbol{m} = \partial_t\boldsymbol{n} = 0, \quad \partial_t\tau = 0. \tag{7.28}$$

The additional conditions

$$\partial_t \rho = 0, \quad \partial_t \gamma = 0 \tag{7.29}$$

then lead to

$$\partial_t \gamma = 0, \quad \partial_t \boldsymbol{t} = 0, \quad \partial_t \boldsymbol{b} = 0 \tag{7.30}$$

where (4.9), (4.11), (4.12), (5.4), (5.5), and (7.26) were taken into account. Thus, the system of Eqs. (7.17) and (7.20) admits a static principal congruence of edge dislocations defined by the conditions (7.26) and (7.29).

Let us take as the second example, the congruence of prismatic edge dislocations of zero torsion defined by (5.16)–(5.23). In this case, $\vartheta = -\pi/2$ in (7.1), and (7.17) reduces to

$$\partial_t \kappa = 0, \quad \partial_{\boldsymbol{l}} \omega = \zeta \kappa, \quad \partial_{\boldsymbol{l}} \zeta = -\omega \kappa. \tag{7.31}$$

We can rewrite (7.31) in the form

$$\partial_{\boldsymbol{l}} w = i \kappa w, \quad w = \zeta + i \omega, \quad \partial_t \kappa = 0 \tag{7.32}$$

where the following condition should be fulfilled:

$$\partial_{\boldsymbol{l}} |w| = 0, \quad |w|^2 = \zeta^2 + \omega^2 \neq 0. \tag{7.33}$$

It follows that if

$$\kappa = \sqrt{2} \partial_{\boldsymbol{l}} \theta > 0, \quad \partial_t \boldsymbol{l} = 0, \quad \partial_t \theta = 0, \tag{7.34}$$

then the complex function

$$w = w_0 e^{i \sqrt{2} \theta} \tag{7.35}$$

where w_0 is a real constant, satisfies (7.32) and (7.33), and (5.21) holds. Moreover, according to (5.19), (5.20), (7.16), and (7.34), the conditions of (7.28) are satisfied. The additional conditions of (7.29) then lead, according to (4.9), (4.11), (4.12), (4.20), (5.17), and (5.19), to the formulas of (7.30). Thus, the system of Eqs. (7.17) and the conditions (4.20) and (7.29) enable us to consider the static congruence of prismatic edge dislocations of zero torsion defined by (5.17)–(5.19) and (5.22).

8. FINAL REMARKS

Let as consider the static congruence of prismatic edge dislocations of zero torsion (Sections 5–7). It is the congruence of dislocation lines as intersections of two orthogonal families of surfaces: the crystal surfaces (on which the dislocations are located) and the slip surfaces (in which the dislocations can move) (Section 5). The crystal surfaces are umbilical minimal varieties (Sections 3 and 5) and thus these are *totally geodesic* surfaces in the (time-independent) equidistant material Riemannian space $\mathcal{B}_g = (\mathcal{B}, \boldsymbol{g})$ (Eisenhart, 1964). This means, among others, that normals to these surfaces are parallel in the enveloping material space (Eisenhart, 1964), that is,

$$\nabla^g \boldsymbol{m} = 0, \tag{8.1}$$

where, according to (5.17), (5.19), (5.22), and (5.23), the field $\boldsymbol{m}$ of local Burgers vector directions covers with the second normal $\boldsymbol{e}_3$ of the congruence and the local Burgers vector $\boldsymbol{b} = b_g \boldsymbol{m}$ is tangent to the slip surfaces. The totally geodesic (crystal) surfaces are an evident generalization of (crystal) planes of Euclidean 3-space (Eisenhart, 1964).

It is known that a dislocation of zero torsion lying in a Euclidean plane experiences a *static straightening force* per unit length of dislocation acting against the direction of curvature vector $\boldsymbol{\kappa}$ and tending to straighten the line (Hull and Bacon, 1984), that is, the force $\boldsymbol{f}$ such that [see (6.1)]

$$\boldsymbol{f} = -S\boldsymbol{\kappa} = -S\kappa \boldsymbol{e}_2, \quad \boldsymbol{f} \cdot \boldsymbol{m} = \boldsymbol{f} \cdot \boldsymbol{l} = 0,$$
$$S > 0, \quad \kappa > 0; \quad [\boldsymbol{e}_2] = [\kappa] = \text{cm}^{-1}, \tag{8.2}$$

where κ is the curvature and $\boldsymbol{e}_2$ is the principal normal of the congruence. If the relation (8.2) is assumed to be valid for the considered congruence of prismatic edge dislocations of zero torsion lying in the totally geodesic crystal surfaces in $\mathcal{B}_g$, then

$$f_g = ||\boldsymbol{f}||_g = S\kappa, \quad [f_g] = \text{kg cm}^{-1}, \quad [S] = \text{kg}. \tag{8.3}$$

The dislocation line will only remain curved if there is a shear stress that produces a force on the dislocation needed to maintain its curvature κ (Hull and Bacon, 1984). So, let $\boldsymbol{T}$ be a symmetric stress tensor considered as acting in the material Riemannian space $\mathcal{B}_g$ (Trzęsowski, 1997, 2000), that is, $\boldsymbol{T}$ can be interpreted as an *internal* stress tensor dependent on the distribution of dislocations and secondary point defects. Let $T = \boldsymbol{mTn}$ denote the field of *shear stresses* resolved in the direction $\boldsymbol{m}$ of the local Burgers vector $\boldsymbol{b}$ of the congruence. The shear stresses T act in the oriented local slip planes $\pi_{\boldsymbol{n}}(\boldsymbol{l}, \boldsymbol{m})$ (Section 2) and the static straightening force $\boldsymbol{f}$ is normal to these planes. We assume, generalizing the statement of Hull and Bacon (1984), that a dislocation line of the strength b_g will be in *local equilibrium* in its curved position when

$$T = \frac{f_g}{b_g}, \quad [b_g] = \text{cm}, \quad [T] = \text{kg cm}^{-2}. \tag{8.4}$$

Substituting f_g from (8.3), we obtain

$$S = \frac{T b_g}{\kappa}. \tag{8.5}$$

The quantity S of (8.5) has units energy per unit length and thus the dislocation line has a *line tension* that is analogous to the surface tension of a soap bubble or liquid

(Hull and Bacon, 1984). Note that the formula (8.5) generalizes the expression of line tension of a curved dislocation lying in a Euclidean plane (see Hull and Bacon, 1984).

The strength b_g of the considered dislocations can be written, according to (5.18), in terms of scalar characteristics γ and ρ of the continuous distribution of dislocations:

$$b_g = \frac{2\gamma}{\rho}. \tag{8.6}$$

The formulas (8.5) and (8.6) lead to the following expression of shear stresses required to bend dislocation lines of the congruence:

$$T = E_d \rho \kappa, \quad E_d = \frac{S}{2\gamma} \tag{8.7}$$

where $E_d > 0$ has units of energy, that is, $[E_d] = \text{kg cm}$. This means that the considered congruence of prismatic edge dislocations is endowed with a finite self-energy function E_d.

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